A SPECIAL MATHEMATICAL MODEL FOR THE STUDY OF ELECTRICAL CIRCUITS COMPONENTS BY THE METHOD OF DIFFERENTIAL INCLUSIONS

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Abstract: Many information technologies for numerical analysis of electrical circuit parameters have been developed. The peculiarity of these methods is their applicability not only to DC but also AC circuits, not only to linear but also nonlinear circuits. This paper introduces the special mathematical model for rectifier circuits using semiconductor diodes. In previous studies, the rectifier circuit has been modelled by differential inclusions. However, the inclusions cannot be solved by the applied programs of information technology. In this paper, the new model writing by differential equation with a special right hand side is studied to replace the classical model. The main result of this is presented in a theorem that gives the correctness of the new model. The usefulness of this study is illustrated via a concrete example.

Key words: classical model, special mathematical model, normal cone, diode, rectifier.

1. INTRODUCTION AND RELATED WORK

A feature of modern methods of analysis and synthesis of electrical circuits is the combination of the stages of approximation, implementation and optimization, while functions for which optimal schemes of the implemented system are known by any criteria are used to approximate the specified characteristics, and in the synthesis process the values of the parameters of the elements are selected to ensure the required accuracy of reproduction of the specified characteristics.

Many information technologies for numerical analysis of electrical circuit parameters have been developed. The peculiarity of these methods is their applicability not only to DC but also AC circuits, not only to linear but also nonlinear circuits. Examples of such systems and processes are transmitter systems [1], energy minimization and task deadline aware workload scheduling [2], in which there is a need to analyze and optimize both events occurring in hardware and processes modeled by the theory of electrical circuits.

As we know, mathematics is the tool for describing changes in each domain as dynamic systems, through which one can indicate their characteristics [3-7]. One of the problems that attracts attention is to study by mathematical modelling an operation of rectifier circuits [6-8]. Most electrical installations use direct current, but the power source is alternating current. Therefore, rectifiers are very important, indispensable and widely used in the electrical industry. The positive elements in the rectifier circuit are semiconductor diodes. A semiconductor diode is a type of semiconductor device that allows the current to flow through it in a single direction: from the anode to the cathode and without reverse direction [8]. According [8] the model describes the operation of rectifier circuit is presented by differential inclusions that is defined as

$$\dot{x} \in f(t,x) - N_Q(x) \tag{1}$$

where $N_Q(x)$ is a cone of special type.

Both existence and uniqueness of solutions of the Cauchy problem are showed in [9]. For this, a solution $x = x(t)$ of (1) is an absolutely continuous function which satisfies (1) for almost all $t \in [t_0, T]$.

2. PROBLEM STATEMENT

A specific problem of rectifier circuit modelled by differential inclusions as example can be found in the section 4.

Obviously, we see that differential inclusions are usually more difficult and complicated than differential equations. In this paper we will study the mathematical model writing by the following differential equation with a special right hand side

$$\dot{x} = P\left(f(t,x), N_Q^* x\right) \tag{2}$$

where $P\left(f(t,x), N_Q^* x\right)$ is the projection of the vector $f(t,x)$ on the adjoin cone of the cone $N_Q(x)$.

A solution of equation (2) is called a locally absolutely continuous function $x = x(t)$ satisfying (2) for almost all $t \in [t_0, T]$ (see [10]).

In the paper, we will study that the model (2) can be substituted for the model writing by the differential inclusion (1). This is the main objective of the paper.

3. MAIN RESULT

To understand model (1) and model (2) as well, we first study the cone $N_Q(x)$ of special type. Let $Q \subset \mathbb{R}^n$ be closed, convex set and $x \in Q$, we are considered the following set

$$N_Q(x) = \{z \in \mathbb{R}^n : (z, \varsigma - x) \le 0,\ \forall \varsigma \in Q\}, \tag{3}$$

It is easy to see that the set $N_Q(x)$ will be a cone in $\mathbb{R}^n$, which is called the *normal cone* to Q at x. The adjoin cone to $N_Q(x)$ is defined by

$$N_Q^*(x) = \{z \in \mathbb{R}^n : (z, \gamma) \le 0,\ \forall \gamma \in N_Q(x)\}. \tag{4}$$

Furthermore, an element $\overline{y} \in Q$ is called point of best approximation to y in Q and denoted by $\overline{y} = P(y, Q)$ if

$$\|y - \overline{y}\| = \inf \{\|y - z\|,\ \forall z \in Q\}. \tag{5}$$

At that, $\overline{y} = P(y, Q)$ exists and is unique for any $y \in \mathbb{R}^n$ and is called the *projection* of $y \in \mathbb{R}^n$ onto Q (see [10]).

Lemma 1. *Let* $\varepsilon > 0$ *small enough,* $0 \le s < \varepsilon$ *and let* $Q \in \mathbb{R}^n$ *be closed and convex. Then the following holds:*

$$x(t+s) \in Q \Rightarrow \dot{x}_+(t) \in N_Q^*(x) \tag{6}$$

$$x(t-s) \in Q \Rightarrow -\dot{x}_-(t) \in N_Q^*(x). \tag{7}$$

Proof. We have

$$\dot{x}_+(t) = \lim_{\xi \to 0+} \frac{x(t+\xi) - x(t)}{\xi}.$$

Moreover, by virtue of (3) we get $(x(t+\xi) - x(t), v) \le 0$ for all $v \in N_Q(x(t))$. From this, we obtain $x(t+\xi) - x(t) \in N_Q^*(x(t))$ (see (4)). Consequently, $\dot{x}_+(t) \in N_Q^*(x)$. Therefore, the condition (3) is proved. Similarly, we can prove the condition (4) is also true. The lemma 1 is completely proved.

Lemma 2. *Let* $Q \in \mathbb{R}^n$ *be closed and convex and* $x \in Q$. *For each* $z \in \mathbb{R}^n$ *the following equation holds*

$$z = \eta + \tau, \tag{8}$$

where $\eta = P(z, N_Q(x)), \tau = P(z, N_Q^*(x))$ *and* $(\eta, \tau) = 0$.

Proof. At first, from definitions (3), (4) and (5) we check that

$$(z - \eta, a - \eta) \le 0 \text{ for all } a \in N_Q(x), \tag{9}$$

$$(z - \eta, b - \eta) \le 0 \text{ for all } b \in N_Q^*(x). \tag{10}$$

Indeed, we denote a function $\varphi(t) = ta + (1-t)\eta,\ t \in [0,1]$. By virtue of (3) and $\eta, a \in N_Q(x)$, we get

$$\left(\varphi(t), c - x\right) = t\left(a, c - x\right) + (1-t)\left(\eta, c - x\right) \le 0,\ t \in [0,1],$$

for all $c \in Q$, hence $\varphi(t) \in N_Q(x)$. Now, let us

$$\psi(t) = \left\| z - \varphi(t) \right\|^2,\ t \in [0,1].$$

From this and (5) it follows $\psi(t) \ge \left\| z - \eta \right\|,\ t \in [0,1]$ and the function $\psi(t)$ has the smallest value at $t = 0$, so $\dot{\psi}(0) \ge 0$. Moreover, we have

$$\dot{\psi}(t) = 2\left(z - \varphi(t), -\dot{\varphi}(t)\right) = 2\left(z - ta - (1-t)\eta, \eta - a\right),\ t \in [0,1].$$

Consequently,

$$\dot{\psi}(t) = 2\left(z - \varphi(t), -\dot{\varphi}(t)\right) = -2\left(z - \eta,\ a - \eta\right) + 2t\left\| \eta - a \right\|^2,\ t \in [0,1].$$

From here, it implies

$$\dot{\psi}(0) = -2\left(z - \eta,\ a - \eta\right) \ge 0.$$

That is, the inequality (9) is proved. Similarly, we can prove the inequality (10) is also true.

Next, suppose there exists $\overline{\tau}$ such that $\overline{\tau} = z - \eta$. By virtue of (3) we have $0 \in N_Q(x)$. From this and (9) it is easy to see that

$$\left(\overline{\tau}, \eta\right) \le 0 \tag{11}$$

Since $\eta = P\left(z, N_Q(x)\right) \in N_Q(x)$, hence $2\eta \in N_Q(x)$. Then, by virtue of (9) we get

$$\left(\overline{\tau}, \eta\right) \ge 0 \tag{12}$$

From (11) and (12), it easily follows $\left(\overline{\tau}, \eta\right) = 0$. Consequently, $\left(\overline{z}, a\right) \le 0$ for all $a \in N_Q(x)$, since $\left(\overline{\tau}, \eta\right) = \left(z - \eta, a - \eta\right) \le 0,\ \forall a \in N_Q(x)$ (see (3)).

Now, using the definition (4) we obtain $\overline{\tau} \in N_Q^*(x)$. Then,

$$\left(z - \overline{\tau}, b - \overline{\tau}\right) = \left(\eta, b\right) - \left(\eta, \overline{\tau}\right) = \left(\eta, b\right) \le 0,\ \forall b \in N_Q^*(x).$$

Therefore, the inequality $\left\| z - \overline{\tau} \right\| \le \left\| z - b \right\|$ holds for all $b \in N_Q^*(x)$. That is, $\left\| z - \overline{\tau} \right\|$ is the smallest length from z to the tangent cone $N_Q^*(x)$. From (5) it follows $\overline{\tau} = P\left(z, N_Q^*(x)\right)$.

To finish the proof of Lemma 2, we have to show $\tau = \overline{\tau}$. Since $\tau, \overline{\tau} \in N_Q^*(x)$ and using (10) we obtain that the vectors $\tau, \overline{\tau}$ are satisfied such that

$$\left(z - \tau, \overline{\tau} - \tau\right) \le 0 \text{ and } \left(z - \overline{\tau}, \tau - \overline{\tau}\right) \le 0. \tag{13}$$

$$\left(z - \tau, \overline{\tau} - \tau\right) = -\left(z - \overline{\tau}, \tau - \overline{\tau}\right) + \left(\overline{\tau} - \tau, \overline{\tau} - \tau\right) \ge 0 \tag{14}$$

From (13) and (14) it implies $\tau = \overline{\tau}$. The Lemma is completely 2 proved.

Theorem 1. *The differential inclusion* (1) *and the differential equation* (2) *are equivalent in the sense that they have the same solutions.*

Proof. First, let us $x(t)$ is an absolutely continuous function on some interval $I \subset \mathbb{R}$ which satisfies (1) almost everywhere. From (1) it easily follows that there exists $\eta \in N_Q(x)$ such that $\dot{x}(t) = f\left(t, x(t)\right) - \eta$. Besides, by virtue of the Lemma 1 and (4) we get

$$\begin{cases} \dot{x} = \dot{x}_+ \in N_Q^*(x) \Rightarrow \left(\dot{x}, \eta\right) \le 0 \\ -\dot{x} = -\dot{x}_- \in N_Q^*(x) \Rightarrow \left(\dot{x}, \eta\right) \ge 0. \end{cases}$$

Consequently, $\left(\dot{x}, \eta\right) = 0$. Thus, we have

$$f(t,x) = \dot{x} + \eta,\ \dot{x} \in N_Q^*(x),\ \left(\dot{x}, \eta\right) = 0.$$

As a result, we get $\dot{x} = P\left(f(t,x), N_Q^*(x)\right) = \tau_x f(t,x)$ (see the Lemma 2), and the last equality is precisely (2).

Now, suppose $x(t)$ is an absolutely continuous function on some interval $I \subset \mathbb{R}$ which satisfies (2) almost everywhere. Applying Lemma 2 there is $\eta \in N_Q(x)$ such that

$$\dot{x} = P\left(f(t,x), N_Q^*(x)\right) = f(t,x) - \eta \in f(t,x) - N_Q(x),$$

which shows that $x(t)$ satisfies the differential inclusion (1). The theorem completely proved.

4. AN EXAMPLE

As we already know, most electrical installations use direct current, but the power source is alternating current. Therefore, rectifiers are very important, indispensable and widely used in the electrical industry (see [7-10]). In the section we will research two mathematical models writing by (1) and (2) which describe characteristics of some rectifier circuits using diodes. We consider a rectifier circuit including a source with voltage $E(t)$, resistance R, inductance L, two elements J_1, J_2 and two semiconductor diodes (Figure 1).

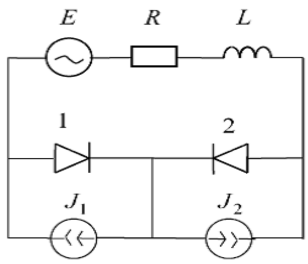

Figure 1. The rectifier circuit

Suppose that i_{D_k} and u_{D_k} are the current and the voltage which passes from the positive pole to the negative pole of the diode $D_k, k=\overline{1,2}$. The characteristics of diodes are represented as follows:

$$i_{D_k} \geq 0;\ u_{D_k} \leq 0;\ i_{D_k} \cdot u_{D_k} = 0. \quad (15)$$

Using Kirchhoff formula, we will have the following equation (see [11]):

$$Ri(t) + Li'(t) + u_{D_2} - u_{D_1} = E(t) \quad (16)$$

and

$$i = i_R = i_L = i_{D_2} - J_2 = -i_{D_1} + J_1, \quad (17)$$

with given constants J_1, J_2.

From (4) it implies $-J_2 \leq i \leq J_1$, so for the correct performance of the circuit we require that $J_1 + J_2 \geq 0$ in order guarantee that $Q = \left[-J_2, J_1\right]$.

Now we distinguish three cases for the position of i in Q:

If $-J_2 < i < J_1$, then $i_{D_1}, i_{D_2} > 0$. From (15) it follows $u_{D_1} = u_{D_2} = 0$.

If $i = -J_2$, then $i_{D_1} > 0, i_{D_2} = 0$. From (15) it follows $u_{D_1} = 0, u_{D_2} \leq 0$.

Finally, if $i = J_1$, then $i_{D_1} = 0, i_{D_2} > 0$. From (15) it follows $u_{D_1} \leq 0, u_{D_2} = 0$.

In any case we get

$$u = u_{D_2} - u_{D_1} \in N_Q i. \quad (18)$$

If we denote $f(t,i) = \frac{E(t)}{L} - \frac{R}{L} i(t)$, the the equation (16) has the form

$$i' = f(t,i) - \frac{u(t)}{L},$$

and using (18) we obtain that also $\frac{u(t)}{L} \in N_Q i$. That is, we get the differential inclusion of the form (1)

$$i' \in f(t,i) - N_Q i,$$

and by virtue of the theorem we also get the model writing by the differential equation of the form (2)

$$i' = P\left(f(t,i), N_Q^* i\right).$$

5. CONCLUSION

A feature of modern methods of analysis and synthesis of electrical circuits is the combination of the stages of approximation, implementation and optimization, while functions for which optimal schemes of the implemented system are known by any criteria are used to approximate the specified characteristics, and in the synthesis process the values of the parameters of the elements are selected to ensure the required accuracy of reproduction of the specified characteristics.

Many information technologies for numerical analysis of electrical circuit parameters have been developed. The peculiarity of these methods is their applicability not only to DC but also AC circuits, not only to linear but also nonlinear circuits. In this article the characteristics of rectifier circuits using semiconductor diodes in control systems were investigated and mathematically modeled using differential switching (1). Nevertheless, it is very difficult to find exact solutions to differential inclusion (1). Therefore, the study of a special model described by differential equation (2) is of high scientific interest. A convenient tool for numerical analysis using information technologies is proposed, for example, with the help of such application programs Mathematica, Matlab, Maple, Electronics Workbench, etc. Based on the results of this study, we hope to get more profound results in further studies and investigate an optimal process for an assembly line of rectifiers in electrical engineering in the control systems. Examples of such systems and processes are transmitter systems [1], energy minimization and task deadline aware workload scheduling [2], in which there is a need to analyze and optimize both events occurring in hardware and processes modelled by the theory of electrical circuits.

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